

# Floydian trajectories for stationary systems: a modification for bound states

M. R. Brown

*Theoretical Physics Research Unit, Birkbeck College, University of London, Malet Street, London WC1E 7HX, England*  
 (February 1, 2008)

The Floydian trajectory method of quantum mechanics and the appearance of microstates of the Schrödinger equation are reviewed and contrasted with the Bohm interpretation of quantum mechanics. The kinematic equation of Floydian trajectories is analysed in detail and a new definition of the variational derivative of kinetic energy with respect to total energy is proposed for which Floydian trajectories have an explicit time dependence with a frequency equal to the beat frequency between adjacent pairs of energy eigenstates in the case of bound systems. In the case of unbound systems, Floydian and Bohmian trajectories are found to be related by a local transformation of time which is determined by the quantum potential.

03.65, 05.45

## I. INTRODUCTION

In a series of papers [1, and references therein], Floyd has developed a trajectory representation of quantum mechanics which, while related to the Bohm interpretation, is distinct in both in its trajectories and its interpretation. Whilst the Floydian method has been strongly advocated and well exemplified by its proponent, in comparison with the Bohm interpretation it seems not to have received as wide critical attention in the literature as perhaps it deserves. Here we attempt to partly redress the balance by briefly reviewing the Floydian method and then critically examining the basis of the kinematic equation of its trajectories. We refer to the Bohm interpretation to stimulate the critical analysis. Floyd has made a detailed comparison of the two approaches in [2].

Two novel aspects of the Floydian method are: 1) the identification of microstates of the Schrödinger equation and 2) a kinematic equation for trajectories which is derived from the Hamilton-Jacobi method of classical mechanics. The method is based upon the direct solution of the Quantum Stationary Hamilton-Jacobi Equation (QSHJE) rather than upon the solution of the corresponding Schrödinger equation and has, thus far, only been developed for 1-dimensional and pseudo-2-dimensional stationary systems. In particular, it has been applied to the finite potential step, the semi-infinite step (in 2-dimensions) [1], the infinite [3] and the finite square potential well [4]. In early work, solutions of the QSHJE were obtained numerically [5] but following the discovery of the form of a general solution in 1-dimension [6], attention has been directed towards study-

ing analytical solutions. The results can be extended to higher dimensions if separation of variables is permitted. However, closed form solutions are not known in the general case and whilst it has been briefly discussed in principle [7], the general case of higher dimensions remains untried in practice.

More recently, the work of Floyd has been set in a wider context by the work of Faraggi and Matone [8] who derive and generalise (to higher dimensions) the QSHJE on the basis of a postulated equivalence principle: that all physical systems are equivalent under coordinate transformations. This connection is of interest, not only because it provides a wider context for the work of Floyd but also because it motivates a connection between the quantum potential and coordinate transformations or deformations of the geometry of space.

Whilst we briefly review the Floydian method and the basis of the appearance of microstates of the Schrödinger equation, our primary focus is upon its trajectory representation. We critically assess the application of the Floydian trajectory method to bound systems. By working in a 3-dimensional coordinate representation we also highlight necessary further assumptions for its application in higher dimensions. This is in contrast to the principally 1-dimensional approach of Floyd.

A point of particular attention, is the appearance of the energy derivative of the kinetic energy in the kinematic equation of Floydian trajectories, especially in quantum systems with discrete energy spectra. Such a derivative is well defined (in fact it is unity) for classical conservative systems, since the transition between one 'stationary' energy state of a given system to another is continuous. However, in bound quantum systems, not only is the energy spectrum discrete (complicating the definition of an energy derivative), but also any attempt to make a transition between the states in order to define a continuous derivative with respect to energy whilst at the same time continuing to satisfy the Schrödinger equation and its boundary conditions faces two problems: 1) states between the eigenstates are time dependent so resulting in a time and space dependent energy derivative 2) there is a multiplicity of such derivatives from which to choose. We explore these problems, (thus far investigated by Floyd only numerically with respect to the harmonic oscillator [9]), for stationary systems in general and examine their relevance to the energy variation of the QSHJE as used by Floyd. These issues do not arise in systems with continuous energy spectra where the Floydian method has been used to investigate barrier reflection and tun-

nelling problems.

The following sections proceed as follows. By way of establishing a comparative base, we first briefly review the equations of the Bohm trajectory method for the general case and subsequently for stationary systems. After briefly reviewing the work of Faraggi and Matone, we then review the Floydian method and the appearance of microstates contrasting the approach with that of Bohm. Deferring until later the kinematic equation of Floydian trajectories, we then derive, for its use, the variational derivative of the kinetic energy with respect to the total energy for stationary systems revealing its space-time dependence and its ambiguity. Taking a more general approach than Floyd, we then derive the kinematic equation of a Floydian trajectory from the variational derivative of the action with respect to the total energy, described by Floyd as the epoch  $\tau$ . We then use our results to demonstrate that Bohmian and Floydian trajectories are related by different assumptions about  $\tau$  and we are led to the idea of time deformation by the quantum potential. After briefly comparing applications to different types of potential, we finally draw together our results in the conclusions.

## II. THE BOHM TRAJECTORY METHOD

In the Bohm approach [10] the quantum Hamilton-Jacobi equation is derived, along with the equation for conservation of probability, from the Schrödinger equation and its complex conjugate by using the polar form of the wave function  $\psi = R(q, t) \exp(i \frac{S(q, t)}{\hbar})$ . For stationary systems,  $R = R(q)$  and  $S(q, t) = W(q) - Et$ , where  $W(q)$  is Hamilton's characteristic function. In the Bohmian method, one obtains  $S(q, t)$  directly from the wave function solution of the Schrödinger equation. The trajectory kinematics are derived from a classical form of the probability current density identified in the probability conservation equation. This method is therefore rooted in the probabilistic interpretation of the wave function.

In the coordinate representation, the substitution  $\psi = R(q, t) \exp(i \frac{S(q, t)}{\hbar})$  in the Schrödinger equation yields:

$$\frac{1}{2m}(\nabla S)^2 + V(q) + Q(q) = -\frac{\partial S}{\partial t} \quad (1)$$

and

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left( \frac{R^2}{m} \nabla S \right) = 0 \quad (2)$$

where

$$Q = -\frac{\hbar^2}{2mR} \nabla^2 R$$

is the quantum potential. From the probability conservation equation (2), one identifies the probability current

density  $j = \frac{R^2}{m} \nabla S$ . Using the classical form  $j = \rho u$  (where  $\rho$  is the density and  $u$  the velocity of a flow) one then identifies  $\dot{q} = \frac{1}{m} \nabla S$  as the streamline velocity of the probability density current which upon integration yields a trajectory. If the density of such trajectories is at any time  $R^2$ , then equation (2) ensures that it is always so. We also observe from equation (1), that with the momentum defined by  $p = \nabla S$ , this equation is the Hamilton-Jacobi equation but for the quantum potential  $Q(q)$ . One may therefore consistently assign  $p = m\dot{q}$ . The latter assignment is strictly true only for conservative systems in a cartesian coordinate representation.

In the case of stationary systems,  $S(q, t) = W(q) - Et$  and so the above equations reduce to

$$\frac{1}{2m}(\nabla W)^2 + V(q) + Q(q) = E \quad (3)$$

and

$$\nabla \cdot \left( \frac{R^2}{m} \nabla W \right) = 0 \quad (4)$$

with the trajectory equation

$$\dot{q} = \frac{1}{m} \nabla W. \quad (5)$$

From the above, we see that for a stationary system of a given energy  $E$ , a Bohm trajectory is defined by a square integrable solution of the Schrödinger equation (to secure the concept of probability) and an initial condition  $x_0$ . Moreover, owing to the single-valuedness of the wave function, Bohm trajectories for a given eigenstate do not cross. For stationary systems with real wave functions, Bohm trajectories are stationary; a point of concern to Einstein who observed that, in the classical  $\hbar \rightarrow 0$  limit, the classical motion is not obtained [11], p243. Floyd [12] has observed that the use of a trigonometric (rather than complex polar) ansatz resolves this artifact. We do not pursue this further here.

## III. THE QUANTUM STATIONARY HAMILTON-JACOBI EQUATION

### A. The quantum stationary Hamilton-Jacobi equation via an equivalence principle

More recently than Floyd's early work, Faraggi and Matone [13–19,8] have independently derived (and generalised to higher dimensions) the QSHJE from a postulated equivalence principle: that all physical systems are equivalent under coordinate transformations. This work extends the implications of Floyd's work on the QSHJE to a connection between quantum mechanics and gravity [8]. In the following, we summarise those key points of the work of Matone et al. that have a bearing on our work here:

- A key corollary of the equivalence postulate is that there must exist a trivialising coordinate transformation  $q \rightarrow q_0$ , defined by the relation between the respective Hamilton's characteristic functions (reduced actions)  $W(q) = W_0(q_0)$ , which reduces any system to that of the free particle [15]:

$$W(q) \rightarrow W_0(q_0)$$

where  $\mathcal{W} = V(q) - E$  and  $\mathcal{W}_0 \equiv 0$ .

- The quantum potential term in the QSHJE is a direct consequence of the postulated universal existence of the trivialising coordinate transformation. In particular, the absence of the quantum potential (as in classical mechanics) prohibits the existence of such a transformation to a frame in which a system is 'at rest'. The existence of the quantum potential therefore enables the removal of this privileged frame.
- The quantum potential is expressed in terms of the Schwarzian derivative of the characteristic function or reduced action  $W_\mu$  (see section III C) which must therefore never be a constant in order for the trivialising map to exist. To guarantee this, a general solution of the Schrödinger equation should be expressed in the form  $\psi = R_\mu(A \exp(iW_\mu/\hbar) + B \exp(-iW_\mu/\hbar))$  rather than in the Bohmian form  $\psi = R \exp(iW/\hbar)$  for real  $R$ ,  $W$ ,  $R_\mu$ , and  $W_\mu$  with  $A$  and  $B$  as complex constants. In the case of real wave functions, this implies that  $A$  and  $B$  must be of equal magnitude and, modulo an integer factor  $2\pi$ , of opposite signed phase so that a real wave function is the sum of a Bohmian polar form and its complex conjugate. (Here and hereafter, we distinguish the characteristic function,  $W_\mu$ , as a direct solution of the QSHJE from the the reduced action,  $W$ , appearing in the phase of the wave function, as used in the Bohm formulation.)
- A general principle is that a general solution of the Schrödinger equation is always has two independent solutions. In general, they are not square integrable. However, under the special conditions for eigenfunctions, one and only one of them may be a square-integrable function on the real line. The Copenhagen interpretation selects the latter solution in order that the wave function be interpreted as a probability amplitude.
- In the case of the QSHJE, a condition for the existence of the Schwarzian derivative is that the total energy  $E$  must take a value such that the QSHJE corresponds to a Schrödinger equation having a wave function that is a square-integrable function on the real line ie  $E$  is an eigenvalue of the corresponding Schrödinger equation. Thus, in solving

the QSHJE, quantisation arises independently of the Copenhagen interpretation demand for square-integrable solutions.

- Time does not appear explicitly in the QSHJE and so the time parameterisation is introduced, after the solutions of the QSHJE are obtained, in the form of trajectories defined using the Floydian ansatz  $t - \tau = \partial W_\mu / \partial E$ . In classical mechanics this is equivalent to the kinematic relation  $p = m\dot{q}$  in which time parameterisation is implicit.

The work of Faraggi et al. draws upon the requirement for satisfactory classical limits (expressed as  $\hbar \rightarrow 0$ ) and through its insistence upon a non-constant reduced action, claims to have side-stepped Einstein's objection that the Bohm interpretation does not have the correct classical limit (in the aforementioned sense) for bound systems such as the quantum harmonic oscillator. We also see that the quantum potential derived from the equivalence principle and used in Floyd's trajectory method does not necessarily equal that of the Bohm interpretation, for the latter is based upon the polar decomposition of only square integrable wave-functions whereas the former are based on the direct solution of the QSHJE and correspond to more general solutions of the Schrödinger equation.

In the next section we focus upon the specific treatment of the QSHJE by Floyd.

## B. Overview of the Floyd method

The Floydian approach [5,20,6,12,7,4,2] originated from an alternative ansatz to the Bohm polar form for solutions of the Schrödinger equation, which is nevertheless consistent with the second Bohm (probability conservation) equation (4). Substituted in the Schrödinger equations, the Floyd ansatz results in a single third order quantum stationary Hamilton-Jacobi (QSHJE) equation for the quantum form of Hamilton's characteristic function  $W_\mu(q)$ . For a given energy eigenvalue of the Schrödinger equation,  $W_\mu(q)$  is not a unique solution of the QSHJE and this leads to the idea that the square integrable wave function for bound states is not exhaustive but has associated microstates. As seen in the previous section, this conclusion has been more generally substantiated by the work of Farragi and Matone. Whilst the Floydian ansatz [21,2] is consistent with the equation for the conservation of probability, its trajectories are not distributed in accordance with the density of the wave function as in the case of the Bohm interpretation. Indeed, unlike Bohmian trajectories, the density of Floydian trajectories would subsequently not be that of the wave function. Rather, Floyd claims that the necessity of the probabilistic interpretation of the wave function

in orthodox quantum mechanics arises from the incomplete specification of a particular microstate. Moreover, each trajectory corresponds to an individual microstate of the Schrödinger equation but is nevertheless sufficient to specify the corresponding wave function [1]. We defer to a later section the derivation of the kinematic equation for Floydian trajectories and here more explicitly outline the work on the quantum stationary Hamilton-Jacobi equation and its solutions in one dimension.

### C. The quantum stationary Hamilton-Jacobi equation and its solutions

The quantum stationary Hamilton-Jacobi equation (QSHJE) is given in one dimension  $q$  by [7,2]

$$\frac{(W'_\mu)^2}{2m} + V - E = -\frac{\hbar^2}{4m}\langle W_\mu; q \rangle \quad (6)$$

where  $W'_\mu = \frac{\partial W_\mu}{\partial q}$  is the momentum conjugate to  $q$  and  $\langle W_\mu; q \rangle$  is the Schwarzian derivative of  $W_\mu$  with respect to  $q$ . The Schwarzian derivative,

$$\langle W_\mu; q \rangle = \frac{W'''_\mu}{W'_\mu} - \frac{3}{2} \left( \frac{W''_\mu}{W'_\mu} \right)^2,$$

which is unaffected by the sign of  $W_\mu$ , means that the QSHJE is a third-order nonlinear differential equation. The left hand side of equation (6) replicates the classical Hamilton-Jacobi equation, whereas the right hand side embodies the quantum effects in the Schwarzian derivative. Comparing equation (6) with equation (3) one may identify the quantum potential term as

$$Q = \frac{\hbar^2}{4m}\langle W_\mu; q \rangle. \quad (7)$$

However, we recognise that the Bohm quantum potential is not always equal to the Schwarzian derivative form. The latter is only guaranteed when  $W_\mu = W$ . Equation (6) can be obtained from the Bohm formulation for stationary systems by recognising that, in one-dimension, equation (4) has a solution  $R = (Cm)/\sqrt{|W'|}$ , where  $C$  is a constant. (Note that this is not generally the case in higher dimensions). Substituting for  $R$  in equation (3) one obtains the form of equation (6). This approach corresponds to substituting  $\psi = \exp(\pm iW_\mu/\hbar)/\sqrt{|W'_\mu|}$  (or a linear combination of the latter) into the Schrödinger equation. Floyd discovered that the general solution,  $W'_\mu$ , of the QSHJE is given by

$$W'_\mu = (2m)^{1/2}(a\phi^2 + b\theta^2 + c\phi\theta)^{-1} \quad (8)$$

where  $[a, b, c]$  is a set of real coefficients such that  $a, b > 0$ , and  $(\phi, \theta)$  must be a pair of (normalised) independent solutions of the stationary Schrödinger equation,

$-\hbar^2\psi''/(2m) + (V - E)\psi = 0$  for given energy  $E$ . (Neither of the the solutions  $(\phi, \theta)$  is necessarily square integrable though, as mentioned in section III A, in the case of eigenstates of bound systems one and only one is.) For given  $E$ ,  $W_\mu$  is therefore prescribed by the three constants  $[a, b, c]$ . Floyd has observed [12] that these constants of the motion may alternatively be represented by the set  $[x_0, \dot{x}_0, \ddot{x}_0]$  of trajectory initial conditions. This is to be contrasted with the set  $[x_0]$  required in classical mechanics to specify  $W_\mu$  for a given  $E$  and is a reflection of the third order nature of the QSHJE. Given the time parameterisation provided by the kinematic equation of the Floydian trajectories, each trajectory therefore specifies a unique microstate  $W_\mu$  and vice-versa. Floydian trajectories for a given eigenstate are therefore correspondingly more various than Bohm trajectories. Indeed, unlike Bohm trajectories, Floydian trajectories (for different microstates) of an eigenstates may cross.

How do the microstates arise? Floyd has shown [7] that for energy eigenstates of stationary bound systems in one dimension (for which the wave function is real), equation (6) has singular values at precisely the locations where the boundary values are applied. This is because  $W'_\mu \rightarrow 0$  as  $x \rightarrow \pm\infty$  since, whilst one of the independent solutions of Schrödinger equation is square-integrable, the other must be unbounded at  $x \rightarrow \pm\infty$  regardless of  $(a, b, c)$  given their constraints. Thus, boundary conditions  $W'_\mu(x \rightarrow \infty) = 0$  admit non-unique solutions for  $W'_\mu$  and so yield an infinity of trajectories. This conclusion also applies to semi-bound systems (eg potential barriers where the total energy is less than the barrier height). We observe that the balance of energy between the kinetic energy and quantum potential terms will differ between the microstates of an eigenstate of such a bound system. Floyd [7] also provides evidence to suggest that an uncountable number of microstates is also possible for bound states in dimensions greater than one, though does not demonstrate that this is the case.

For unbound systems, in which the (complex) solution of the Schrödinger equation is an initial value problem, the initial conditions determine a unique solution to equation (6) for  $W'_\mu$ . In this case, the same solution is obtained in the Bohm interpretation ie  $W_\mu = W$ . Thus, there are no microstates in this case and the trajectories, for given initial conditions, are unique. However, because their kinematic equations differ, the Floydian and Bohmian trajectories follow different paths through space-time [12].

### D. General form of the wave function used for Floydian trajectories

Given a solution  $W_\mu$  of the QSHJE for a stationary state of energy  $E$ , the corresponding wave function  $\Psi$

may written in the general bi-polar form described in section III A:

$$\Psi = \psi_\mu^+ + \psi_\mu^- \quad (9)$$

where

$$\psi_\mu^\pm = A_\mu^\pm R_\mu \exp\left(\frac{\pm iW_\mu}{\hbar}\right) \exp\left(-i\frac{Et}{\hbar}\right). \quad (10)$$

$A_\mu^\pm$  are complex constants and  $R_\mu^2 \propto 1/|\nabla W_\mu|$  as discussed in section III C. This form is motivated by the fact that the QSHJE is invariant under a sign change of  $W_\mu$ , so that both  $\psi_\mu^\pm$  and any linear combination of them lead to the same QSHJE and continuity equation when substituted into the Schrödinger equation. (For a given stationary state,  $\Psi$  may, of course, also be expressed in the form of a synthesized wave [2] using the Bohmian ansatz  $\Psi = \tilde{R}_\mu \exp\left(\frac{i\tilde{W}_\mu}{\hbar}\right) \exp\left(-i\frac{Et}{\hbar}\right)$ . Selection of the proper form depends on the physical situation.) The case  $A_\mu^- = 0$  can represent the complex eigenfunctions of unbound stationary systems whereas  $A_\mu^- = A_\mu^{+*}$  can represent the real eigenfunctions of bound systems. Moreover, in both instances the equivalence principle requirement that  $W_\mu \neq \text{constant}$  is fulfilled. Thus, the general wave function  $\Psi$  comprises two waves,  $\psi_\mu^\pm$ , of identical form and energy, travelling in opposite directions but scaled by different different complex amplitudes  $A_\mu^\pm$  respectively. We refer to these two waves as the positive and negative running waves in correspondence with the sign prefix of  $W_\mu$  in their phase factors. An important aspect for the next section is that the momentum and energy in the QSHJE can be derived from either of such a pair of running waves using  $p_\mu = \nabla S_\mu$  and  $E = -\partial S_\mu / \partial t$  where  $S_\mu = W_\mu - Et$ .

#### IV. THE TOTAL ENERGY VARIATIONAL DERIVATIVE OF KINETIC ENERGY FOR STATIONARY SYSTEMS

We now investigate the variation of kinetic energy with respect to the total energy for stationary systems. This variational dependence is used in the definition of the kinematic equation for Floydian trajectories and in their connection with Bohmian trajectories. Though Floyd actually worked with the variation of the quantum potential [5] with respect to the total energy, it is simply related to the variation of kinetic energy with respect to total energy. Here we find it more convenient to work with the latter.

The use of an energy derivative implies a continuous spectrum of admissible energy eigenvalues. Indeed, Floyd provides examples of Floydian trajectories (using the energy derivative of the quantum potential) for systems with continuous energy spectra for which such a derivative is easily defined [12]. In his work on bound states, in

which microstates of the Schrödinger equation are identified, Floyd [5] gives examples of microstate trajectories in mechanical phase space for the quantum harmonic oscillator, based on a power series solution (in the space variable) for the modified potential (classical plus quantum potential) which has energy depend coefficients. For such systems, with discrete spectra, the concept of the energy derivative is unnatural, for the energy is not obviously a continuous variable. Indeed, in section III A we saw that a condition for the existence of the Schwarzian derivative was that the energy correspond to an eigenstate of the Schrödinger equation. It is therefore necessary to modify the methods in [5] to take account of the time dependence which an energy variation between eigenstates might induce. Thus, instead of Floyd's  $\partial Q/\partial E$ , we adopt the notation  $\delta Q/\delta E$  described as a variational derivative with respect to energy.

In the following sub-sections, we investigate the energy variational derivative of the kinetic energy  $\delta T/\delta E$  from the standpoint of stationary quantum systems with distinct energy eigenstates. The derivative (justifiably described as partial) for continuous energy spectra is then obtained as a limiting case of the discrete spectrum result.

##### A. Stationary state variation

Let  $\Psi_i$  be a stationary solution of the Schrödinger equation

$$i\hbar \frac{\partial \Psi_i}{\partial t} = H\Psi_i \quad (11)$$

for a given potential  $V(q)$  and set  $\mathcal{B}$  of boundary (or initial) conditions. So that we remain within the principles of quantum mechanics and stay with a description of the same physical system, we impose upon the varied wave function,  $\Psi = \Psi_i + \delta\Psi$ , the condition that it also satisfies the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi, \quad (12)$$

for the same potential  $V(q)$  and conditions  $\mathcal{B}$ . We also impose the further condition that the varied wave function preserve the space normalisation of the original wave function but accept that the varied wave function will not necessarily be a stationary solution of the Schrödinger equation.

Expressing  $\Psi_i$  and  $\Psi$  in the bi-polar form of section III D, we obtain their respective Hamilton-Jacobi equations whose difference may be written as

$$\delta \left( \frac{1}{2m} (\nabla S_\mu)^2 \right) + \delta Q_\mu = \delta \left( -\frac{\partial S_\mu}{\partial t} \right). \quad (13)$$

where  $S_\mu = \pm W_\mu - Et$ . This equation may be expressed in the simple form

$$\delta T_\mu + \delta Q_\mu = \delta E \quad (14)$$

where, by identification of terms in equation (13),  $T_\mu$  is the kinetic energy and  $E$  the total energy. This is a general equation for variations satisfying the constraints of the Schrödinger equation regardless of boundary conditions and normalisation. Thus, under such general conditions, we may have  $E = E(q, t)$ . In the following, the variational derivative  $\delta T_\mu / \delta E$  is obtained from the ratio of the variations  $\delta T_\mu$  and  $\delta E$  with respect to a particular form of infinitesimal variation of the wave function  $\delta\Psi$  which satisfies (1) the Schrödinger equation, (2) the conditions  $\mathcal{B}$  and (3) preserves the normalisation of the wave function.

All three conditions on the varied wave function  $\Psi$  are satisfied if it is expressed in the form

$$\Psi = \cos(\theta)\Psi_i + \sin(\theta)\Psi_j \quad (15)$$

where  $\theta$  is the parameter of the variation and  $\Psi_j$  is another normalised stationary solution (eigenfunction) of the Schrödinger equation for the same potential  $V(q)$  and conditions  $\mathcal{B}$ . The orthonormality of  $\Psi_i$  and  $\Psi_j$  guarantees the  $\theta$ -independence of the normalisation of  $\Psi$ . That  $\Psi$  is a linear combination of solutions of the Schrödinger equation for the conditions  $\mathcal{B}$  means that it also is a solution of the same. In principle, one can construct any solution as a linear combination of any number of eigenfunctions. However, the objective in this case is to study the relative variation of the kinetic and total energies under an infinitesimal change of the wave function from one eigenstate towards another whether or not they have a continuous energy spectrum; hence the form of equation (15). Whilst this analysis could be carried out for any pair of eigenfunctions  $\Psi_i$  and  $\Psi_j$ , here, we define  $\Psi_i$  and  $\Psi_j$  to be non-degenerate eigenstates corresponding respectively to adjacent different eigenvalues  $E_i$  and  $E_j$  in the ordered energy spectrum, so that  $E_j > E_i$ . In this way, we remove ambiguity from the definition of the variational derivative with respect to total energy and bring it into correspondence with the partial derivative  $\partial T / \partial E$  in classical conservative systems.

As discussed in section III D,  $\Psi_i$ ,  $\Psi_j$  and thus  $\Psi$  may be decomposed into positive and negative running waves so that  $\Psi = \psi_\mu^+ + \psi_\mu^-$  with

$$\psi_\mu^\pm = \psi_{\mu i}^\pm \cos \theta + \psi_{\mu j}^\pm \sin \theta \quad (16)$$

where

$$\psi_{\mu k}^\pm = A_k^\pm R_{\mu k} \exp\left(\frac{\pm i W_{\mu k}}{\hbar}\right) \exp\left(-i \frac{E_k t}{\hbar}\right) \quad (17)$$

are themselves the positive and negative running waves for the microstate solution  $W_{\mu k}$  of the QSHJE for the  $k$ th eigenstate with eigenvalue  $E_k$ .

## B. Analysis of kinetic energy variational derivative

In this section, for the sake of notational clarity, we initially drop the microstate suffix  $\mu$  on the variables  $p$ ,  $S$ ,  $T$ ,  $\psi$  and  $Q$  and re-introduce them in the final result.

Consider a variation of the wave function parameterised by  $\delta\theta$ . From equations (13) and (14), the change in the kinetic energy is

$$\delta T = \frac{p}{m} \frac{\partial p}{\partial \theta} \delta\theta \quad (18)$$

where  $p = \nabla S$  is the conjugate momentum. Similarly, the change in the total energy is

$$\delta E = \frac{\partial E}{\partial \theta} \delta\theta \quad (19)$$

where  $E = -\partial S / \partial t$  is the total energy. Thus, in this work, the variational derivative of the kinetic energy with respect to the total energy is defined by

$$\frac{\delta T}{\delta E} = \frac{1}{m} \frac{p \cdot \partial p / \partial \theta}{\partial E / \partial \theta}. \quad (20)$$

In both the latter equations the partial derivatives with respect to  $\theta$  are evaluated at  $\theta = 0$  ie in the unvaried state. In passing, we note from equation (14) that  $\delta Q / \delta E = 1 - \delta T / \delta E$ .

The subsequent analysis proceeds by respectively expressing the conjugate momentum and total energy in terms of one of the running waves (we choose the positive wave) of the varied wave function (see equation (16)), so that

$$\begin{aligned} p &= \nabla S = \Im\left(\frac{\hbar}{\psi} \nabla \psi\right) \\ E &= -\frac{\partial S}{\partial t} = -\Im\left(\frac{\hbar}{\psi} \frac{\partial \psi}{\partial t}\right) \end{aligned} \quad (21)$$

in which, and hereafter, the  $\pm$  superscript has been dropped. Therefore,

$$\begin{aligned} \frac{\partial p}{\partial \theta} &= \hbar \Im\left[-\frac{1}{\psi} \frac{\partial \psi}{\partial \theta} \frac{1}{\psi} \nabla \psi + \frac{1}{\psi} \frac{\partial}{\partial \theta} \nabla \psi\right] \\ \frac{\partial E}{\partial \theta} &= -\hbar \Im\left[-\frac{1}{\psi} \frac{\partial \psi}{\partial \theta} \frac{1}{\psi} \frac{\partial \psi}{\partial t} + \frac{1}{\psi} \frac{\partial}{\partial \theta} \frac{\partial \psi}{\partial t}\right] \end{aligned} \quad (22)$$

where the partial derivative terms with respect to  $\theta$  at  $\theta = 0$  are computed from equation (16) (without the superscripts) as

$$\begin{aligned} \left(\frac{1}{\psi} \frac{\partial \psi}{\partial \theta}\right)_{\theta=0} &= \frac{\psi_j}{\psi_i} \\ \left(\frac{1}{\psi} \frac{\partial}{\partial \theta} \nabla \psi\right)_{\theta=0} &= \frac{1}{\psi_i} \nabla \psi_j \\ \left(\frac{1}{\psi} \frac{\partial}{\partial \theta} \frac{\partial \psi}{\partial t}\right)_{\theta=0} &= \frac{1}{\psi_i} \frac{\partial \psi_j}{\partial t}. \end{aligned} \quad (23)$$

With these results equations (22) become

$$\begin{aligned} \left(\frac{\partial p}{\partial \theta}\right)_{ij} &= \hbar \Im \left[ -\frac{\psi_j}{\psi_i} \left( \frac{\nabla \psi_j}{\psi_j} - \frac{\nabla \psi_i}{\psi_i} \right) \right] \\ \left(\frac{\partial E}{\partial \theta}\right)_{ij} &= -\hbar \Im \left[ -\frac{\psi_j}{\psi_i} \left( \frac{1}{\psi_j} \frac{\partial \psi_j}{\partial t} - \frac{1}{\psi_i} \frac{\partial \psi_i}{\partial t} \right) \right], \end{aligned} \quad (24)$$

in which the partial derivatives with respect to  $\theta$  at  $\theta = 0$  have been labelled by the suffices  $ij$  to show that they are evaluated at the eigenstate  $i$  ‘in the direction of’ the eigenstate  $j$ . Expressing the running wave for the  $k$ th eigenstate in the form of equation (17) and re-introducing the microstate suffix  $\mu$ , equations (23) may be used in equation (20) to finally give

$$\begin{aligned} \left(\frac{\delta T}{\delta E}\right)_{\mu ij} &= \frac{\nabla S_{\mu i}}{m \Delta E_{ji}} . \\ \left[ \nabla (\Delta S_{\mu ji}) + \hbar \tan \left( \frac{\Delta S_{\mu ji}}{\hbar} + \Delta \alpha_{\mu ji} \right) \nabla \ln \left( \frac{R_{\mu j}}{R_{\mu i}} \right) \right], \end{aligned} \quad (25)$$

where  $\Delta E_{ji} = E_j - E_i$ ,  $\Delta S_{\mu ji} = S_{\mu j} - S_{\mu i}$  and  $S_{\mu k}(q, t) = W_{\mu k}(q) - E_k t$ .  $\Delta \alpha_{\mu ji} = \alpha_{\mu j} - \alpha_{\mu i}$  is the difference between the phases of the running wave scale factors of the  $j$ th and  $i$ th eigentsates for microstate  $\mu$ , where  $A_{\mu k} = |A_{\mu k}| \exp(i\alpha_{\mu k})$ .

We recall that this analysis was done for the positive running wave but observe that the result (25) is invariant to a change of sign prefix to  $W_\mu$  (for the negative running wave) as expected if the scale factor phases also undergo a corresponding sign change. The latter condition does indeed occur when the eigenstates are real as in the case of bound states. (See section III D.) Equation (25) clearly shows that  $(\delta T / \delta E)_{\mu ij}$  may be explicitly time dependent on account of the ‘beating’ of the different frequencies of the states  $i$  and  $j$ .

### C. The kinetic energy variational derivative for unbound systems

In the limit of unbound systems (including barriers), which have continuous energy spectra,  $\Delta S_{\mu ji}$  and  $\Delta E_{ji}$  become infinitesimals and  $R_{\mu j} \rightarrow R_{\mu i}$  so equation (25) becomes

$$\left(\frac{\delta T}{\delta E}\right)_{\mu ij} \rightarrow \left(\frac{\delta T}{\delta E}\right)_\mu = \frac{\nabla S_\mu}{m} \cdot \nabla \left( \frac{\partial S_\mu}{\partial E} \right), \quad (26)$$

where the right hand side is identical to the classical form.

In the case of a free system in a constant potential, the spectrum is continuous and  $R$  is a constant so that the quantum potential is zero. We may then replace  $\delta T / \delta E$  by  $\partial T / \partial E$  in equation (26) and obtain precisely the classical result. Clearly, for unbound systems, the energy derivative of the kinetic energy has no explicit time dependence.

### D. The kinetic energy variational derivative for bound systems

In the case of stationary bound systems with discrete spectra and real eigenfunctions, equation (25) may be applied to any one of the corresponding Floydian microstates of the Schrödinger equation. Referring to equation (8) and the text immediately thereafter, a microstate may be identified by the set  $[q_0, \dot{q}_0, \ddot{q}_0, E_i]$  of trajectory initial conditions and the energy eigenvalue. Thus, to determine from equation (25) the kinetic energy variational derivative for the microstate of the  $i$ th eigenstate defined by the initial conditions  $[q_0, \dot{q}_0, \ddot{q}_0]$ , we may define  $\Delta W_{\mu ji}(q) = W_\mu(q, [q_0, \dot{q}_0, \ddot{q}_0, E_j]) - W_\mu(q, [q_0, \dot{q}_0, \ddot{q}_0, E_i])$  holding the trajectory initial conditions constant. For such microstates, the conjugate momentum  $\nabla S_{\mu i}$  is non-zero and the second term within the square bracket of equation (25) generates the explicit time dependence of  $(\delta T / \delta E)_{\mu ij}$  with a frequency equal to the difference between the frequencies of the states  $i$  and  $j$ . We therefore observe that for a given set of initial conditions there is a family of explicitly time dependent trajectories corresponding to the family of discrete eigenstates. The explicit time dependence of such bound state microstate trajectories does not appear in the work of Floyd.

## V. A TRAJECTORY EQUATION FROM THE ENERGY VARIATIONAL DERIVATIVE OF THE ACTION

### A. The nature of the energy variational derivative

Here we study the basis of the Floydian trajectory method as defined for stationary systems. The approach, in the augmented domain of the motion  $(q, t, E)$ , is based upon the variation of the action with respect to changes in the total energy  $E$  which are out of the constant energy plane of the motion. As such, it is restricted to stationary systems. We emphasise that the kinematic equations so obtained are for Floydian trajectories in the  $(q, t)$  domain.

Here we make a point of clarification. Carroll [21] connects the evaluation of the total energy derivative of the action with the Legendre transformation from the  $(q, t)$  domain to the  $(q, E)$  domain. In the case of stationary systems, such a transformation corresponds to the substitution of one independent variable,  $t$ , by another,  $E$ , which is a single valued constant of the motion. Thus, the trajectory in  $(q, E)$  space would trivially lie on a fixed energy plane labelled by  $E$ . (In the case of non-stationary systems the Legendre transformation is well defined and the motion is not restricted to such energy planes.) We emphasise that the Legendre transformation of a given motion between the  $(q, t)$  and  $(q, E)$  domains is to be

clearly distinguished from the following study of variations of the motion in the  $(q, t, E)$  domain. We now proceed with the latter. We once again drop the microstate suffix  $\mu$  and the eigenstate suffices  $ij$ , using them only when clarity demands.

### B. The energy variational derivative of the action and a trajectory kinematic equation

Consider the energy variational derivative of the (quantum or classical) action  $S(q, t, E)$  which, following Floyd, we define as the epoch

$$\tau = -\frac{\delta S}{\delta E}. \quad (27)$$

In using the variational derivative  $\delta S/\delta E$  we are consistent with section IV but differ from Floyd, who uses the partial derivative. The use of the latter is justified only in the case of stationary classical and quantum systems which have a continuum of states of spatially uniform energy. Our use of the variational derivative acknowledges that in quantum mechanics the variation  $\delta E$  may necessarily be a function of  $q$  and  $t$  in order for the Schrödinger equation and its boundary or initial conditions to remain satisfied.

We suspend Floyd's assumption that  $\tau$  is a constant and examine its time derivative along a trajectory in  $(q, t)$  space in the energy plane labelled by  $E$ . Thus, from equation (27),

$$\frac{d\tau}{dt} = -\frac{d}{dt}\left(\frac{\delta S}{\delta E}\right) = -\left[\frac{\partial}{\partial t}\left(\frac{\delta S}{\delta E}\right) + \nabla\left(\frac{\delta S}{\delta E}\right) \cdot \dot{q}\right]. \quad (28)$$

Changing the order of  $\partial t$  and  $\delta E$  and using the Hamilton-Jacobi equation with  $E = -\partial S/\partial t$  (see equation (14)), equation (28) becomes

$$\nabla\left(\frac{\delta S}{\delta E}\right) \cdot \frac{\dot{q}}{(1-d\tau/dt)} = 1 \quad (29)$$

where  $\dot{q}$  is the total derivative of the trajectory position with respect to time.

A further equation involving  $\delta S/\delta E$  is obtained from the energy variational derivative of the kinetic energy  $T = (\nabla S)^2/2m$ . Thus, after changing the order of  $\delta$  and  $\nabla$  as applied to  $S$ ,

$$\nabla\left(\frac{\delta S}{\delta E}\right) \cdot \frac{\nabla S}{m(\delta T/\delta E)} = 1, \quad (30)$$

where the  $ij$  subscripts, dropped from  $(\delta T/\delta E)_{ij}$ , are understood. Comparison of equations (29) and (30) shows that equation (29) is satisfied by

$$\dot{q} = \frac{\nabla S}{m} \frac{(1-d\tau/dt)}{(\delta T/\delta E)}. \quad (31)$$

Whilst this solution is consistent with the one-dimensional case, for which it is the only solution, in higher dimensions it is not the only solution of equation (29). For an  $n$ -dimensional system, equation (29) defines  $\dot{q}/(1-d\tau/dt)$  as a member of an infinite set,  $\mathcal{Q}_{n-1}$ , of vectors passing from the origin and ending on an  $(n-1)$ -dimensional plane which has a normal vector  $\nabla(\delta S/\delta E)$  and which is located at a distance  $1/|(\nabla(\delta S/\delta E))|$  from the origin. For such a system, equation (30) shows that  $\nabla S/(m(\delta T/\delta E)) \in \mathcal{Q}_{n-1}$  also. By choosing  $\dot{q}/(1-d\tau/dt)$  to be this particular member of  $\mathcal{Q}_{n-1}$ , we obtain equation (31) and achieve consistency with the 1-dimensional case. Thus, in all except the 1-dimensional case, the form of the trajectory equation (31) is suggested, rather than prescribed, by the above variational analysis.

We note that in this derivation, no use has been made of the probability conservation equation (4). Thus, unlike Bohmian trajectories, the trajectories defined by equation (31) do not have a natural probabilistic interpretation. We discuss this matter further in the next section.

## VI. CONNECTIONS BETWEEN CLASSICAL, FLOYDIAN AND BOHMIAN TRAJECTORIES

The Floydian and Bohmian trajectories are connected by different assumptions about the time dependence of the parameter  $\tau$ . In classical mechanics, this parameter is demonstrably a constant with respect to the time  $t$ . This is not the case in quantum mechanics.

### A. Classical systems

For conservative classical systems,  $\delta T/\delta E = 1$  and in cartesian coordinates  $m\dot{q} = \nabla S$  so equation (31) implies that  $d\tau/dt = 0$  ie  $\tau$  is indeed a constant without assumption. On the other hand, the converse is not true: Assuming that  $d\tau/dt = 0$  is a necessary though not sufficient condition for  $m\dot{q} = \nabla S$  except in the case of one space dimension, as discussed in section V B.

### B. The kinematic equation of Floydian trajectories

The kinematic equation of Floydian trajectories arises from the *assumption* that  $d\tau/dt = 0$ . Thus, with this assumption equation (31) becomes

$$\dot{q} = \frac{\nabla S}{m} \frac{1}{(\delta T/\delta E)}, \quad (32)$$

in which we may use the expression for  $(\delta T/\delta E)_{ij}$  given in equation (25).

If we explicitly write the action in the stationary form  $S = W - Et$  so that  $\nabla S = \nabla W$  and recall from equation (14) that  $\delta T/\delta E = 1 - \delta Q/\delta E$ , then equation (32) becomes

$$\dot{q} = \frac{\nabla W}{m} \frac{1}{(1 - \delta Q/\delta E)}, \quad (33)$$

which is precisely the form used by Floyd [5]. For (Floydian) microstate trajectories defined by this kinematic equation,  $\dot{q} \neq \frac{p}{m}$  where  $p = \nabla W$  is the conjugate momentum. This is different from the kinematic relation for both classical and Bohm trajectories, the latter applying directly to the reduced action  $W$  of the wave function and not microstate solutions  $W_\mu$ .

In passing, we note that equation (33) represents a deformation of the classical kinematic relation between conjugate momentum and velocity. In making a parallel with relativity, Faraggi and Matone [18] capture this deformation in the form of a 'quantum mass field'

$$m_Q = m(1 - \delta Q/\delta E) = m\delta T/\delta E,$$

so restoring the classical form  $\dot{q} = p/m_Q$ . Only for microstate solutions of bound states, for which  $\delta T/\delta E$  is not uniformly zero, is the 'quantum field mass' a meaningful concept. The multiplicity of such microstates implies an equal multiplicity of 'quantum mass fields'.

### C. The kinematic equation of Bohmian trajectories and quantum time

The kinematic equation of Bohmian trajectories is derived from the conservation of probability equation and the classical form for the current density. This unambiguously implies that  $\dot{q} = \frac{1}{m}\nabla S$  in the coordinate representation so coinciding with the classical kinematic relation. (A different kinematic equation occurs in other representations of the Bohm formulation [22].) The classical and Bohm kinematic relations therefore coincide. Thus, rather than being derived from an assumption regarding  $\tau$ , the Bohm kinematic equation may be used to define the derivative  $d\tau/dt$ . In the coordinate representation, equation (31) implies that

$$\frac{d\tau}{dt} = \frac{\delta Q}{\delta E} \quad (34)$$

showing that, in general, the 'epoch'  $\tau$  is not a constant for a Bohmian trajectory. Defining the 'quantum time' as  $t_Q = t - \tau$ , we find that

$$\frac{dt_Q}{dt} = (1 - \delta Q/\delta E) = \delta T/\delta E, \quad (35)$$

along the trajectory. This equation thus suggests that the quantum potential acts to deform time. The kinematic equation for Bohm trajectories with respect to  $t_Q$  is then

$$\frac{dq}{dt_Q} = \frac{\nabla S}{m(\delta T/\delta E)} \quad (36)$$

which, for stationary systems, has the same form as equation (32) for Floydian trajectories with respect to the classical  $t$ .

We observe from equation (35) that for conservative classical systems,  $t_Q$  and  $t$  advance at the same rate and, at most, differ by an arbitrary constant shift  $\tau$ . This is consistent with section VI A and the idea that a classical particle has neither internal energy nor extension in space-time.

In the case of unbound quantum systems [12], there is a single microstate which can be compared with a Bohmian trajectory. In this instance, the above equation (36) suggests that, in a region of interference where the quantum potential is non-zero, the differing forms of the Floydian and Bohmian trajectories can be related by a deformation of time which is determined by the quantum potential.

For stationary quantum systems the square integrable real wave functions (such as in bound states), have  $\nabla W = \nabla S = 0$  and  $\delta Q/\delta E = 1$ , so neither Bohm trajectories nor quantum time  $t_Q$  evolve with respect to the classical time  $t$ ; they stand still. In fact for such systems, equation (36) is not well defined, this being a symptom the fact that the basis of Bohm trajectories is inconsistent with that of the microstates of bound state wave functions.

### D. Applications to different potentials

To illuminate the general results of the previous sections, we briefly discuss their application to different forms of potential.

In the case of unbound potentials, results are obtained for both Floydian and Bohmian trajectories and the Floydian  $W_\mu$  and the Bohmian  $W$  are the same. For constant potentials classical results are obtained. For the step-potential which is less than the total energy  $E$  [12], there are points in the interference domain where the energy derivative of the kinetic energy passes through zero and changes sign so yielding Floydian trajectories that have infinite positive and negative velocities. This is in distinction to the positive Bohmian velocities in the same domain.

In the case of bound potentials [5] Floydian and Bohmian trajectories differ not only because of differences in the kinematic equations of their trajectories but also because  $W_\mu$  and  $W$ , from which the are respectively derived, differ both in form and multiplicity. For bound potentials, Floydian trajectories are found, in this work, to have explicit time dependence which is at the beat frequency between adjacent eigenstates whereas Bohmian trajectories have zero velocity. Though not explicitly addressed here, we would also expect this principle of ex-

plicit time dependence to apply to finite bound potentials such as the finite square well [1] where  $E < V_0$  (the well depth).

## VII. CONCLUSIONS

The emphasis of this paper has been technical rather than philosophical and interpretational. Having briefly reviewed the Floydian trajectory method in quantum mechanics, we focused attention upon basis of the kinematic equation of Floydian trajectories.

Firstly, motivated by concerns over the definition of the total energy derivative of the kinetic energy and thus the quantum potential, especially in the case of the quantum systems with discrete energy spectra, we derived an explicitly time dependent form for the energy derivative of the kinetic energy for use in the kinematic equation of Floydian trajectories. This explicit periodic time dependence, common to all microstates of a given eigenstate and a modification of the kinematic equation used by Floyd, is at the beat frequency between two adjacent eigenstates of a bound system and is therefore itself quantised. In the limiting case of systems with continuous spectra, this modified kinematic equation becomes the original Floydian form. Our approach is based on the use a total energy variational derivative of the kinetic energy derived from a variation of the wave function constrained to be in the space spanned by the eigenfunctions of the Schrödinger equation. This is in contrast to the approach of Floyd who, treating the total energy as a parameter of each microstate, directly computes the alternative total energy partial derivative of the quantum potential without consideration of the temporal interference with other eigenstates induced by a variation in energy.

Secondly, we have examined the derivation of the kinematic equation of Floydian trajectories from the energy derivative of the action (the epoch) by first setting aside Floyd's assumption that the epoch is a constant and working without the restriction to one dimension. We find that in general, whilst the magnitude of the velocity of the Floydian trajectory is well defined, its direction is not; the exception being in one-dimension or in systems with uncoupled degrees of freedom. This arbitrariness of direction is overcome by assuming the multi-dimensional form of the kinematic equation to be consistent with that in one dimension. We show that in classical mechanics the epoch is a constant without assumption, whereas assuming it is so in quantum mechanics leads the kinematic equation of Floydian trajectories. On the other hand, assuming the Bohm kinematic equation we find the epoch not to be constant and are led to the idea of the deformation of time by the quantum potential. In the case of unbound systems, for which only is this a consistent interpretation, we find that Floydian and Bohmian trajectories can be related by a time deformation.

Further work is required to investigate the application of the explicitly time dependent form of the total energy derivative of the kinetic energy to the various bounded potentials already analysed by Floyd.

## ACKNOWLEDGEMENT

The author gratefully acknowledges the helpful comments provided by M. Matone and by E. R. Floyd in response to an earlier version of this paper. Some points raised by E. R. Floyd are the subject of further investigation.

- [1] E. R. Floyd, Found. Phys. Lett. 13(3), 235-251, (2000).
- [2] E. R. Floyd, preprint quant-ph/0009070, (2000).
- [3] E. R. Floyd, Int. J. Mod. Phys. A, 14(7), 111-1124, (1999).
- [4] E. R. Floyd, Int. J. Mod. Phys. A, 15(9), 1363-1378, (2000).
- [5] E. R. Floyd, Phys. Rev. D, 26(6), 1339-1347, (1986).
- [6] E. R. Floyd, Phys. Rev. D, 34(10), 3246-3249, (1986).
- [7] E. R. Floyd, Found. Phys. Lett., 9(5), 489-493, (1996).
- [8] M. Matone, preprint hep-th/0005274, (2000).
- [9] E. R. Floyd, Phys. Rev. D, 25, 1547, (1982).
- [10] D. Bohm and B. J. Hiley, The Undivided Universe, Routledge (1993).
- [11] P. R. Holland, The Quantum Theory of Motion, Cambridge University Press, (1993).
- [12] E. R. Floyd, Physics Essays, 7(2), 135-144, (1994).
- [13] A. E. Faraggi and M. Matone, Physics Letters B, 445(1-2), 78-81, (1999).
- [14] A. E. Faraggi and M. Matone, Physics Letters B, 450(1-3), 34-40, (1999).
- [15] A. E. Faraggi and M. Matone, Physics Letters B, 455(3-4), 357-365, (1999).
- [16] A. E. Faraggi and M. Matone, Physics Letters A, 249(3), 180-190, (1998).
- [17] A. E. Faraggi and M. Matone, Physics Letters B, 437(3-4), 369-380, (1998).
- [18] A. E. Faraggi and M. Matone, Int. J. Mod. Phys. A, 15(13), 1869-2017, (2000).
- [19] G. Bertoli, A. E. Faraggi and M. Matone, Class. Quant. Grav., 17(19), 3965-4005, (2000).
- [20] E. R. Floyd, Phys. Rev. D, 29(8), 1842-1844, (1984).
- [21] R. Carroll, J. Can. Phys., 77(4), 319-325, (1999).
- [22] M. R. Brown and B. J. Hiley, preprint quant-ph/0005026, (2000).